MATH20132 Calculus of Several Variables. 2020-21

Solutions to Problems 9 Lagrange's Method

1. For $\mathbf{x} \in \mathbb{R}^2$ let $f(\mathbf{x}) = x^2 - 3xy + y^2 - 5x + 5y$

i. Find the critical values of $f(\mathbf{x})$ in $\mathbb{R}^2$,

ii. Find the critical values of $f(\mathbf{x})$ restricted to the parametric curve $(t^2, t^3)^T$, $t \in \mathbb{R}$,

iii. Find the critical values of $f(\mathbf{x})$ restricted to the level set $x + 6y = 6$ (use Lagrange's method).

Solution i The critical values of $f(\mathbf{x})$ in $\mathbb{R}^2$ are the solutions of $\nabla f(\mathbf{x}) = \mathbf{0}$. The two components of the gradient vector give $2x - 3y - 5 = 0$ and $-3x + 2y + 5 = 0$. So the critical point in $\mathbb{R}^2$ is $(1, -1)^T$.

ii. For the critical points of $f(\mathbf{x}) : \mathbf{x} = (t^2, t^3)^T, t \in \mathbb{R}$ look for the critical points of $f\left((t^2, t^3)^T\right) : t \in \mathbb{R}$, i.e. when the gradient vector is zero.

For a function of one variable the gradient vector has one component, the derivative of

$$f\left((t^2, t^3)^T\right) = t^4 - 3t^5 + t^6 - 5t^2 + 5t^3,$$

which is $6t^5 - 15t^4 + 4t^3 + 15t^2 - 10t$. This factors as

$$t(t-1)(t+1)\left(6t^2 - 15t + 10\right)$$

(The square can be completed in the quadratic factor as $6(t - 5/4)^2 + 5/8$ which shows that it is never zero and so cannot be factored further.)

Thus there are critical points when $t = 0, 1$ and -1, i.e. at points

$$(0, 0)^T, \ (1, 1)^T \ \text{ and } \ (1, -1)^T.$$

iii. For the critical points of $f(\mathbf{x}) : x + 6y = 6$ we use Lagrange's method. So if $g(\mathbf{x}) = x + 6y - 6$, we try to solve $\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$ along with $x + 6y = 6$. The co-ordinates of the gradient vectors give the system

$$2x - 3y - 5 = \lambda \quad \text{and} \quad 3x + 2y + 5 = 6\lambda.$$

Solve for x and y :

$$x = 1 - 4\lambda \quad \text{and} \quad y = -1 - 3\lambda.$$

Yet we require $x + 6y = 6$, i.e. $(1 - 4\lambda) - 6(1 + 3\lambda) = 6$, which leads to $\lambda = -1/2$. Hence the only critical point is $(3, 1/2)^T$.

Note we did not need to use Lagrange's method, we could instead have substituted $x = -6y + 6$ in $f(\mathbf{x})$ and looked for the turning points $f_y(\mathbf{x}) = 0$.

The point of the question is that a function $f(\mathbf{x})$, $\mathbf{x} \in S \subseteq \mathbb{R}^n$ may have different critical points depending on the set S. Also, if S is given parametrically as the image of $\mathbf{g}(\mathbf{u})$, $\mathbf{u} \in \mathbb{R}^m$, we look for critical points of $f(\mathbf{g}(\mathbf{u}))$. Thus Lagrange's method is only applied when S is a level set.

2. i. Find the minimum value of $3x^2 + 3y^2 + z^2$ subject to the condition $x + y + z = 1$.

ii. Find the maximum and minimum values of xy subject to the condition $x^2 + y^2 = 1$.

iii. Find the minimum and maximum values of xy^2 subject to the condition $x^2/a^2 + y^2/b^2 = 1$ (where a and b are positive constants).

Solution i. Let $f(\mathbf{x}) = 3x^2 + 3y^2 + z^2$, and $g(\mathbf{x}) = x + y + z - 1$, $\mathbf{x} \in \mathbb{R}^3$. We wish to find $\min \{f(\mathbf{x}) : g(\mathbf{x}) = 0\}$.

The set $\{\mathbf{x} : g(\mathbf{x}) = 0\}$ is a level set and, to be a surface, the Jacobian of g has to be full rank. Yet g is scalar-valued so this is equivalent to demanding that the gradient of g is non-zero. Here $\nabla g(\mathbf{x}) = (1, 1, 1)^T$ for all $\mathbf{x}$ and so is non-zero for all $\mathbf{x}$ and we can apply the method of Lagrange multipliers. This gives the equation $\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$ for some λ along with $g(\mathbf{x}) = 0$.

Write these equations as

$$\begin{aligned} (6x,\, 6y,\, 2z) &= \lambda (1,\, 1,\, 1)\,, \\ x + y + z &= 1. \end{aligned}$$

The first gives $y = x$, $z = 3x$. In the second this gives $x = 1/5$ in which case $y = 1/5$ and $z = 3/5$. (That $\lambda = 6/5$ is true but of no interest.) Hence

$$\mathbf{a} = (1/5,\, 1/5,\, 3/5)^T$$

is an extremal point of $f(\mathbf{x})$ restricted to $g(\mathbf{x}) = 0$. At this point $f(\mathbf{a}) = 3/25 + 3/25 + 9/25 = 3/5$.

The set of $\mathbf{x} : x + y + z = 1$ is closed but not bounded, so we cannot immediately say that $f(\mathbf{x})$ attains it's lower bound at $\mathbf{a}$. You could argue by first restricting to the box $|x|, |y|, |z| \le 1$. We now have a closed and bounded region on which $f(\mathbf{x})$ **will** attain it's lower bound. When you look for this point you will either find $\mathbf{a}$ or a point on the boundary. But for any point on the boundary or even outside the box, i.e. when we have at least one of $|x| \geq 1$, $|y| \geq 1$ or $|z| \geq 1$, then $f(\mathbf{x}) \geq 1 > f(\mathbf{a})$. Thus $f(\mathbf{a})$ is the minimum value.

ii. Let $f(\mathbf{x}) = xy$, and $g(\mathbf{x}) = x^2 + y^2 - 1$ with $\mathbf{x} \in \mathbb{R}^2$. Here $\nabla g(\mathbf{x}) = (2x, 2y)^T$ which is non-zero for all $\mathbf{x} : g(\mathbf{x}) = 0$. So we can apply the method of Lagrange multipliers. The method gives the equations

$$(y,\, x) = \lambda\,(2x,\, 2y) \quad \text{along with} \quad x^2 + y^2 = 1.$$

From the first $y = 2\lambda x$ and $x = 2\lambda y$ which together gives $x = 4\lambda^2 x$. The solutions of this are **either** $x = 0$ **or** $\lambda = \pm 1/2$.

- If $x = 0$ then $y = 2\lambda x = 0$ but $(0,0)^T$ is not a point satisfying $x^2 + y^2 = 1$.
- If $\lambda = \pm 1/2$ then $y = 2\lambda x = \pm x$. In $x^2 + y^2 = 1$ this gives $x = \pm 1/\sqrt{2}$. Thus we have four solutions

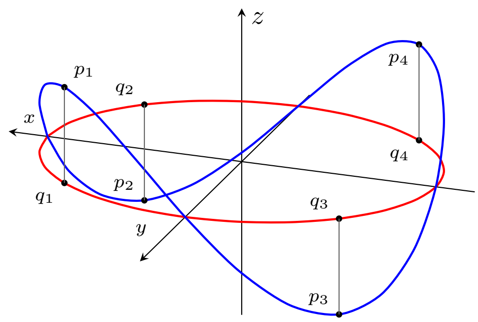
$$\begin{aligned} \mathbf{a}_1 &= \left(1/\sqrt{2},\, 1/\sqrt{2}\right)^T,\ \mathbf{a}_2 = \left(1/\sqrt{2},\, -1/\sqrt{2}\right)^T, \\ \mathbf{a}_3 &= \left(-1/\sqrt{2},\, 1/\sqrt{2}\right)^T,\ \mathbf{a}_4 = \left(-1/\sqrt{2},\, -1/\sqrt{2}\right)^T. \end{aligned}$$

Since the circle $x^2 + y^2 = 1$ is a closed and bounded set the continuous function f must have minimum and maximum values on it. These must occur within the points we have found.

Checking,

- $f(\mathbf{a}_2) = f(\mathbf{a}_3) = -1/2$ is the minimum value,
- $f(\mathbf{a}_1) = f(\mathbf{a}_4) = 1/2$ the maximum.

What we are doing in this problem is finding the points on the blue line with the largest height, i.e. largest value of z, with $(x, y)^T$ restricted to the red circle.



iii. Let $f(\mathbf{x}) = xy^2$ where $\mathbf{x} \in \mathbb{R}^2$, subject to the condition $x^2/a^2 + y^2/b^2 - 1 = 0$. Here $\nabla g(\mathbf{x}) = (2x/a^2, 2y/b^2)^T$ which is non-zero for all $\mathbf{x} : g(\mathbf{x}) = 0$. So we can apply the method of Lagrange multipliers. The method gives the equations

$$(y^2, 2xy) = \lambda \left(\frac{2x}{a^2}, \frac{2y}{b^2} \right) \quad \text{and} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

From $2xy = \lambda y/b^2$ **either** $y = 0$ **or** $x = \lambda/b^2$.

- If $x = \lambda/b^2$ then, combined with $y^2 = 2\lambda x/a^2$, we get $y^2 = 2\lambda^2/a^2b^2$. In $x^2/a^2 + y^2/b^2 = 1$ we get

$$\frac{1}{a^2}\left(\frac{\lambda}{b^2}\right)^2 + \frac{1}{b^2}\frac{2\lambda^2}{a^2b^2} = 1, \text{ i.e. } \lambda = \pm\frac{ab^2}{\sqrt{3}}.$$

  Thus we get four points

$$\mathbf{x} = \left(\frac{\lambda}{b^2}, \pm\sqrt{2}\frac{\lambda}{ab}\right)^T = \left(\pm\frac{a}{\sqrt{3}}, \pm\sqrt{2}\frac{b}{\sqrt{3}}\right)^T.$$

- If $y = 0$ then $x = \pm a$ and we get two more point $(\pm a, 0)^T$.

Since the ellipse $x^2/a^2 + y^2/b^2 = 1$ is closed and bounded a continuous function must have minimum and maximum values. By evaluating the function at the points found above we see that the minimum is $-2ab^2/3\sqrt{3}$, the maximum $2ab^2/3\sqrt{3}$ (given that a is positive.)

Note In both parts ii & iii the problem can be reduced to a problem of one variable:

i. Finding extrema of xy subject to $x^2+y^2=1$ is the same as finding extrema of $\pm x\sqrt{1-x^2}$;

ii Finding extrema of xy^2 subject to $x^2/a^2+y^2/b^2=1$ is the same as finding extrema of $b^2x(1-x^2/a^2)$.

But, if asked to use Lagrange's method, use it!

3 Find points on the circle $(x-2)^2+(y+1)^2=4$ which are a maximum and minimum distance from the origin.

Hint consider the **square** of the distance.

Solution Follow the hint and find the minimum and maximum of the *square* of the distance function from the origin to $(x,y)^T$. So we start by finding the critical points of x^2+y^2 subject to $(x-2)^2+(y+1)^2=4$.

Here $\nabla g(\mathbf{x}) = (2(x-2), 2(y+1))^T \neq \mathbf{0}$ for all $\mathbf{x} : g(\mathbf{x}) = 0$. The method of Lagrange's multipliers then gives

$$2x = 2\lambda(x-2) \text{ and } 2y = 2\lambda(y+1).$$

Rearrange as

$$(\lambda-1)(x-2) = 2 \text{ and } (\lambda-1)(y+1) = -1.$$

Multiply $(x-2)^2+(y+1)^2=4$ by $(\lambda-1)^2$ and substitute in $(\lambda-1)(x-2) = 2$ to get $2^2+(-1)^2 = 4(\lambda-1)^2$. The resulting $4(\lambda-1)^2=5$ has two solutions $\lambda = 1\pm\sqrt{5}/2$. These lead to the points

$$\left(\frac{10+4\sqrt{5}}{5}, -\frac{5+2\sqrt{5}}{5}\right)^T \text{ and } \left(\frac{10-4\sqrt{5}}{5}, -\frac{5-2\sqrt{5}}{5}\right)^T,$$

respectively.

Note this can be checked. Without proof it seems reasonable that if we consider the straight line through the origin and the centre of the circle $(2,-1)^T$, then the circle will intersect this line at the points we need (this would require a proof). The point of the question is that if you need a critical point of the distance you can find a critical point of the *square* of the distance.

4. Find the minimum distance from the point on the x-axis $(a,0)^T \in \mathbb{R}^2$ to the parabola $y^2 = x$.

Solution As in the previous question, consider the *square* of the distance from $(a,0)^T$ to a point $(x,y)^T$ on the parabola, which is $(x-a)^2 + (y-0)^2$.

So, we need to minimise $f(\mathbf{x}) = (x-a)^2 + y^2$ subject to the condition $g(\mathbf{x}) = 0$, $\mathbf{x} \in \mathbb{R}^2$, where $g(\mathbf{x}) = y^2 - x$. Here $\nabla g(\mathbf{x}) = (-1, 2y)^T$ which is never zero so we can apply the method of Lagrange multipliers. This gives

$$\begin{aligned} 2(x-a) &= -\lambda, \\ 2y &= 2\lambda y, \\ y^2 &= x. \end{aligned}$$

From $2y = 2\lambda y$ **either** $y = 0$ **or** $\lambda = 1$.

- If $y = 0$ then $x = y^2 = 0$ too.
- If $\lambda = 1$ then $2(x-a) = -1$, i.e. $x = a - 1/2$ and $y = \pm\sqrt{x} = \pm\sqrt{a - 1/2}$ *provided* $a \geq 1/2$.

So the critical points are $\mathbf{0}$ and, when $a > 1/2$,

$$\mathbf{a}_1 = \left(a - 1/2,\ \sqrt{a - 1/2}\right)^T, \ \mathbf{a}_2 = \left(a - 1/2,\ -\sqrt{a - 1/2}\right)^T.$$

Checking at the points found: $f(\mathbf{0}) = a^2$ and, if $a \geq 1/2$, $f(\mathbf{a}_1) = f(\mathbf{a}_2) = a - 1/4$. Take the positive root to find the distance and we have the minimum distance is

$$\begin{cases} |a| & \text{if } a < 1/2 \\ \sqrt{a - 1/4} & \text{if } a \geq 1/2. \end{cases}$$

The set of $\mathbf{x} : g(\mathbf{x}) = 0$ is closed but not bounded so we need an ad-hoc argument (not given here) to prove that we have, in fact, found the minimum values.

5. Find the extremal values of $f(\mathbf{x}) = xy + yz$, $\mathbf{x} \in \mathbb{R}^3$ on the level set

$$\begin{aligned} x^2 + y^2 &= 1 \\ yz - x &= 0. \end{aligned}$$

Solution For this we need that the Jacobian of the level set is of full rank. The Jacobian is

$$\begin{pmatrix} 2x & 2y & 0 \\ -1 & z & y \end{pmatrix}.$$

On $x^2 + y^2 = 1$ we cannot have x and y zero simultaneously, so the top row of the Jacobian is never 0. The two rows are possibly linearly dependent if $y = 0$, but then $yz = x$ implies $x = 0$ which we have noted is not possible. Thus the Jacobian matrix is of full rank for all $\mathbf{x}$ in the level set and we can apply the method of Lagrange multipliers.

At extremal values there exist $\lambda, \mu \in \mathbb{R}$ such that

$$\nabla f(\mathbf{x}) = \lambda \nabla \left(x^2 + y^2\right) + \mu \nabla \left(yz - x\right).$$

So we have the system

$$y = \lambda 2x - \mu, \quad x + z = 2y\lambda + \mu z, \quad y = \mu y, \quad x^2 + y^2 = 1 \text{ and } yz = x.$$

From $y = \mu y$ **either** $y = 0$ **or** $\mu = 1$.

- If $y = 0$ the last two conditions become $x^2 = 1$ and $0 = x$ of which there is no solution.
- So $y \neq 0$ and $\mu = 1$, when the system becomes

  $$y = \lambda 2x - 1, \quad x = 2y\lambda, \quad x^2 + y^2 = 1 \text{ and } yz = x.$$

  From the second, $2\lambda = x/y$, which in the first gives $y = x^2/y - 1$. Rearrange so $y^2 + y = x^2 = 1 - y^2$, having used the third equation. Therefore $2y^2 + y - 1 = 0$. This factorises as $(2y - 1)(y + 1) = 0$. The solution $y = 1/2$ gives $x = \pm\sqrt{3}/2$ and $z = \pm\sqrt{3}$. The solution $y = -1$ gives $x = 0 = z$.

Hence, the solutions are

$$\begin{aligned} \mathbf{a}_1 &= (0,\, -1,\, 0)^T, \\ \mathbf{a}_2 &= \left(\sqrt{3}/2,\, 1/2,\, \sqrt{3}\right)^T, \\ \mathbf{a}_3 &= \left(-\sqrt{3}/2,\, 1/2,\, -\sqrt{3}\right)^T. \end{aligned}$$

Calculating f at these points give $f(\mathbf{a}_1) = 0$, $f(\mathbf{a}_2) = 3\sqrt{3}/4$, the maximum value, $f(\mathbf{a}_3) = -3\sqrt{3}/4$, the minimum value.

The set of $\mathbf{x} : g(\mathbf{x}) = 0$ is closed but not bounded so we need an ad-hoc argument (not given here) to prove that we have, in fact, found the extremum values.

6. Find the maximum and minimum values of $4y - 2z$ subject to the conditions $2x - y - z = 2$ and $x^2 + y^2 = 1$.

Solution The level set is closed and bounded. ($x^2+y^2 = 1$ implies $|x|, |y| \leq 1$ while $2x - y - z = 2$ means $|z| = |2x - y - 2| \leq 2|x| + |y| + 2 \leq 5$, by the triangle inequality.) The function $f(\mathbf{x}) = 4y - 2z$ is continuous and so must have maximum and minimum values on the level set.

The Jacobian matrix of the level set is

$$\begin{pmatrix} 2 & -1 & -1 \\ 2x & 2y & 0 \end{pmatrix}.$$

This is not of full-rank only if $x = y = 0$ which, because of $x^2 + y^2 = 1$ does not lie on the level set. So at all points of the level set the Jacobian matrix is of full-rank and we can apply the method of Lagrange multipliers.

At extremal values there exist $\lambda, \mu \in \mathbb{R}$ such that $\nabla f(\mathbf{x}) = \lambda \nabla g^1(\mathbf{x}) + \mu \nabla g^2(\mathbf{x})$. This gives system of equations

$$\begin{aligned} 0 &= 2\lambda + 2\mu x \\ 4 &= -\lambda + 2\mu y \\ -2 &= -\lambda, \end{aligned}$$

along with $2x - y - z = 2$ and $x^2 + y^2 = 1$.

Substituting $\lambda = 2$ into the first two equations gives $\mu y = 3$ and $\mu x = -2$. Then, multiplying $x^2 + y^2 = 1$ by μ gives $\mu^2 = (\mu x)^2 + (\mu y)^2 = 4 + 9$ so $\mu = \pm\sqrt{13}$. Thus

$$x = \mp 2/\sqrt{13}, \quad \text{and} \quad y = \pm 3/\sqrt{13}.$$

Then

$$z = 2x - y - 2 = \mp 4/\sqrt{13} \mp 3/\sqrt{13} - 2 = \mp 7/\sqrt{13} - 2.$$

So the two critical points of f on the surface are

$$\begin{aligned} \mathbf{a}_1 &= \left(2/\sqrt{13}, -3/\sqrt{13}, 7/\sqrt{13} - 2\right)^T, \\ \mathbf{a}_2 &= \left(-2/\sqrt{13}, 3/\sqrt{13}, -7/\sqrt{13} - 2\right)^T. \end{aligned}$$

All that remains are the calculations

$$f(\mathbf{a}_1) = -26/\sqrt{13} + 4 = -2\sqrt{13} + 4,$$
$$f(\mathbf{a}_2) = 2\sqrt{13} + 4.$$

Therefore the maximum value of f on S is $2\sqrt{13}+4$, the minimum $-2\sqrt{13}+4$.

7 Find the minimum distance between a point on the circle in $\mathbb{R}^2$ with the equation $x^2 + y^2 = 1$ and a point on the parabola in $\mathbb{R}^2$ with the equation $y^2 = 2(4 - x)$.

Solution Let $(x, y)^T$ be a point on the circle, so $x^2 + y^2 = 1$, and let $(u, v)^T$ be a point on the parabola, so $v^2 = 2(4 - u)$. Then, as in Question 2, the problem is to minimize the function $f(\mathbf{x}) = (x - u)^2 + (y - v)^2$, where $\mathbf{x} = (x, y, u, v)^T$ (the *square* of the distance between $(x, y)^T$ and $(u, v)^T$) subject to the constraint

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} x^2 + y^2 - 1 \\ v^2 - 2(4 - u) \end{pmatrix} = \mathbf{0}.$$

The Jacobian matrix

$$J\mathbf{g}(\mathbf{x}) = \begin{pmatrix} 2x & 2y & 0 & 0 \\ 0 & 0 & 2 & 2v \end{pmatrix}$$

is not of full rank only if either row is zero. The second row is obviously never zero, the first is if $x = y = 0$ but this does not satisfy $x^2 + y^2 = 1$. Hence we can apply the method of Lagrange multipliers. This gives the equations

$$\nabla f(\mathbf{x}) = \lambda \nabla g^1(\mathbf{x}) + \mu \nabla g^2(\mathbf{x}), \ x^2 + y^2 = 1 \quad \text{and} \quad v^2 = 2(4 - u).$$

The first of these is

$$\begin{pmatrix} 2(x - u) \\ 2(y - v) \\ -2(x - u) \\ -2(y - v) \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 2 \\ 2v \end{pmatrix}.$$

So

$$\begin{aligned} (x - u) &= \lambda x, \qquad (1) \\ (y - v) &= \lambda y, \\ -(x - u) &= \mu, \\ -(y - v) &= \mu v. \end{aligned}$$

There are many ways to solve this system, what follows is just one.

From the last pair $-(x-u)=\mu$ and $-(y-v)=\mu v$ we get $v(x-u)=y-v$. Then from the first pair

$$\lambda y = y - v = v(x-u) = \lambda v x,$$

i.e. $\lambda y = \lambda v x$. Thus **either** $\lambda = 0$ **or** $y = vx$.

- If $\lambda = 0$ then from the first two lines in (6) we have $x = u$ and $y = v$, i.e. $(x, y) = (u, v)$. But this is impossible since the curves $x^2 + y^2 = 1$ and $y^2 = 2(4-x)$ do not intersect. (If they did x would satisfy $1 - x^2 = 2(4-x)$ and you can check this has no real roots.)
- If $y = vx$ then multiply $v(x-u) = y - v$ through by x and use $vx = y$ to get $y(x-u) = x(y-v)$ i.e. $uy = vx = y$. Thus **either** $y = 0$ **or** $u = 1$.

  * If $y = 0$ then from $x^2 + y^2 = 1$, $x = \pm 1$. From $y = vx$, $v = 0$ in which case, from $v^2 = 2(4-u)$, we obtain $u = 4$. So we get the two points

  $$\mathbf{a}_1 = (1, 0, 4, 0)^T \text{ and } \mathbf{a}_2 = (-1, 0, 4, 0)^T.$$

  * If $u = 1$ then, from $v^2 = 2(4-u)$,we obtain $\nu = \pm\sqrt{6}$. Then $y = vx = \pm\sqrt{6}x$. Using $x^2 + y^2 = 1$ we find $x = \pm 1/\sqrt{7}$. Thus we get a further four points

  $$\begin{aligned} \mathbf{a}_3 &= \left(1/\sqrt{7},\ \sqrt{6/7},\ 1,\ \sqrt{6}\right)^T, \\ \mathbf{a}_4 &= \left(1/\sqrt{7},\ -\sqrt{6/7},\ 1,\ -\sqrt{6}\right)^T, \\ \mathbf{a}_5 &= \left(-1/\sqrt{7},\ -\sqrt{6/7},\ 1,\ \sqrt{6}\right)^T, \\ \mathbf{a}_6 &= \left(-1/\sqrt{7},\ \sqrt{6/7},\ 1,\ -\sqrt{6}\right)^T. \end{aligned}$$

  Note that because of $y = vx$ there is not a free choice on the sign of y, it follows from the choices for x and v, thus four points.

Now we are left with the calculations, $f(\mathbf{a}_1) = 9, f(\mathbf{a}_2) = 25$,

$$f(\mathbf{a}_3) = f(\mathbf{a}_4) = 8 - 2\sqrt{7} \quad \text{and} \quad f(\mathbf{a}_6) = f(\mathbf{a}_5) = 8 + 2\sqrt{7}$$

The minimum distance therefore is $8 - 2\sqrt{7}$, approximately 2.70849......

8. An ellipse in $\mathbb{R}^3$ is given by the equations

$$\begin{cases} 2x^2 + y^2 = 4, \\ x + y + z = 0. \end{cases}$$

The intersection of a cylinder with a plane.

Use the method of Lagrange multipliers to find the points on the ellipse which are closest to the y-axis.

(This is a question from the June 2012 examination which turned out to be too difficult! It should be alright away from the pressure of the examination room. When you come to solving a system of equations remember to focus on finding x, y and z, i.e. remove the Lagrange parameters λ and μ as soon as possible.)

Solution Given points $(x, y, z)^T$ on the ellipse and $(0, v, 0)^T$ on the y-axis, the square of their distance apart is $x^2 + (y - v)^2 + z^2$. When this is minimal we must have $y = v$, and so it remains to minimise $f(\mathbf{x}) = x^2 + z^2$, subject to $\mathbf{g}(\mathbf{x}) = \mathbf{0}$ where

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} 2x^2 + y^2 - 4 \\ x + y + z \end{pmatrix}.$$

The level set $\mathbf{x} : \mathbf{g}(\mathbf{x}) = \mathbf{0}$ is closed and bounded. The function f is continuous and so will be bounded and will attain it's bounds.

The Jacobian matrix of $\mathbf{g}$ is

$$J\mathbf{g}(\mathbf{x}) = \begin{pmatrix} 4x & 2y & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

This is not full rank only if $x = y = 0$ but this does not occur in any solution of $\mathbf{g}(\mathbf{x}) = \mathbf{0}$. Hence we can apply the method of Lagrange multipliers and solve

$$\nabla f(\mathbf{x}) = \lambda \nabla g^1(\mathbf{x}) + \mu \nabla g^2(\mathbf{x}) \quad \text{with} \quad \mathbf{x} \in \mathbb{R}^3, \lambda, \mu \in \mathbb{R} \quad \text{and} \quad \mathbf{g}(\mathbf{x}) = \mathbf{0}.$$

This gives the equations

$$\begin{aligned} 2x &= 4\lambda x + \mu, \\ 0 &= 2\lambda y + \mu, \\ 2z &= \mu, \end{aligned}$$

along with $\mathbf{g}(\mathbf{x}) = \mathbf{0}$. Substituting the third equation $\mu = 2z$ into the first two give

$$x = 2\lambda x + z \quad \text{and} \quad \lambda y = -z.$$

Multiply the first of these by y and substitute in the second to get

$$xy = -2zx + zy = -z(2x - y).$$

From $g^2(\mathbf{x}) = 0$ we have $-z = x + y$ so

$$xy = (x + y)(2x - y) = 2x^2 + xy - y^2, \ \text{ i.e. } \ y^2 = 2x^2.$$

From $g^1(\mathbf{x}) = 0$, we have $4 = 2x^2 + y^2$. Combined with $y^2 = 2x^2$ this gives $4 = 4x^2$ so $x = \pm 1$. Then $y = \pm\sqrt{2}$ and z follows from $z = -x - y$.

This leads to four critical points of f restricted to the surface:

$$\begin{aligned} \mathbf{a}_1 &= \left(1, \sqrt{2}, -1 - \sqrt{2}\right)^T, \\ \mathbf{a}_2 &= \left(1, -\sqrt{2}, -1 + \sqrt{2}\right)^T, \\ \mathbf{a}_3 &= \left(-1, \sqrt{2}, 1 - \sqrt{2}\right)^T, \\ \mathbf{a}_4 &= \left(-1, -\sqrt{2}, 1 + \sqrt{2}\right)^T. \end{aligned}$$

Calculating,

$$\begin{aligned} f(\mathbf{a}_1) &= 4 + 2\sqrt{2}, \\ f(\mathbf{a}_2) &= 4 - 2\sqrt{2}, \\ f(\mathbf{a}_3) &= 4 - 2\sqrt{2}, \\ f(\mathbf{a}_4) &= 4 + 2\sqrt{2}. \end{aligned}$$

So $\mathbf{a}_2$ and $\mathbf{a}_3$ are the points on the ellipse closest to the y-axis.

Solutions to Additional Questions

Solutions have not been written up for all of the following.

9 Show that xy has a maximum on the ellipse $9x^2 + 4y^2 = 36$ and find it's value.

Solution The function xy is continuous, the ellipse $9x^2+4y^2 = 36$ is a closed and bounded set. Hence xy is bounded and attains it's bounds.

Lagrange's multipliers gives $y = 18\lambda x$ and $x = 8\lambda y$. Then

$$x = 8\lambda y = 8\lambda\,(18\lambda x)$$

so **either** $x = 0$ **or** $1 = 144\lambda^2$.

- If $x = 0$ then $y = 18\lambda x = 0$. Yet $(0,0)^T$ does not satisfy $9x^2+4y^2 = 36$ so there are no critical points with $x = 0$.
- If $1 = 144\lambda^2$ then $\lambda = \pm 1/12$ and thus $y = \pm 3x/2$. In $9x^2 + 4y^2 = 36$ this gives $18x^2 = 36$ and thus $x = \pm\sqrt{2}$. Hence we have four critical points:

$$\left(\sqrt{2}, 3/\sqrt{2}\right)^T, \left(\sqrt{2}, -3/\sqrt{2}\right)^T, \left(-\sqrt{2}, 3/\sqrt{2}\right)^T \text{ and } \left(-\sqrt{2}, -3/\sqrt{2}\right)^T.$$

The maximal xy will come from critical points with non-zero coordinates of the same sign, i.e. $\left(\sqrt{2}, 3/\sqrt{2}\right)^T$ and $\left(-\sqrt{2}, -3/\sqrt{2}\right)^T$. Hence the maximal value is 3.

10 Find the maximum and minimum values of

$$x^2 + y^2 + z^2 - xy - xz - yz$$

subject to the condition

$$x^2 + y^2 + z^2 - 2x + 2y + 6z + 9 = 0.$$

Solution We can complete the squares so

$$\begin{aligned} 0 &= x^2 + y^2 + z^2 - 2x + 2y + 6z + 9 \\ &= (x-1)^2 + (y+1)^2 + (z+3)^2 - 2. \end{aligned}$$

Thus, geometrically, we are looking for the extrema of

$$f(\mathbf{x}) = x^2 + y^2 + z^2 - xy - xz - yz$$

subject to $\mathbf{x} \in \mathbb{R}^3$ lying on the surface of a sphere, centre $(1, -1, -3)^T$, radius $\sqrt{2}$. The surface of a sphere is closed and bounded. The function f is continuous and so will be bounded and will attain it's bounds.

Let $g(\mathbf{x}) = x^2 + y^2 + z^2 - 2x + 2y + 6z + 9$. Then

$$Jg(\mathbf{x}) = (2x - 2,\ 2y + 2,\ 2z + 6),$$

which is zero only if $\mathbf{x} = (1, -1, -3)$. But since g is not zero at this point $Jg(\mathbf{x})$ is of full rank at $\mathbf{x} : g(\mathbf{x}) = 0$. So we can apply the method of Lagrange multiplies, which requires solving $\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$ for some $\lambda \in \mathbb{R}$. From this we get

$$\begin{aligned} 2x - y - z &= 2\lambda x - 2\lambda, && (2) \\ 2y - x - z &= 2\lambda y + 2\lambda, && (3) \\ 2z - x - y &= 2\lambda z + 6\lambda, \end{aligned}$$

along with $g(\mathbf{x}) = 0$.

Summing the equations above gives $0 = 2\lambda(x + y + z + 3)$. So **either** $\lambda = 0$ **or** $x + y + z + 3 = 0$.

- If $\lambda = 0$ then

  $$\begin{aligned} 2x - y - z &= 0, \\ 2y - x - z &= 0, \\ 2z - x - y &= 0. \end{aligned}$$

  Subtracting the first two gives $x = y$. In the third get $x = y = z$. From $g(\mathbf{x}) = 0$ then get $3x^2 + 6x + 9 = 0$, i.e. $x^2 + 2x + 3 = 0$. But this has no real solutions since $x^2 + 2x + 3 = (x + 1)^2 + 2 \geq 2 > 0$.

- Hence $\lambda \neq 0$ and we must have $x + y + z + 3 = 0$. Rearrange, $z = -3 - x - y$ and substitute into (2) and (3) to get

  $$(2\lambda - 3)\, x = 3 + 2\lambda \quad \text{and} \quad (2\lambda - 3)\, y = 3 - 2\lambda.$$

  Then

  $$(2\lambda - 3)\, z = -3(2\lambda - 3) - x(2\lambda - 3) - y(2\lambda - 3) = -6\lambda + 3.$$

  Substitute into $g(\mathbf{x}) = 0$ to get

  $$0 = -2\frac{4\lambda^2 - 12\lambda - 27}{(2\lambda - 3)^2}$$

The numerator factorises as $(3+2\lambda)(2\lambda-9)$ so we find two solutions $\lambda=-3/2$ and $\lambda=9/2$. Substituted back in we find

$$\mathbf{a}_1=(0,-1,-2)^T \quad \text{and} \quad \mathbf{a}_2=(2,-1,-4)^T.$$

The calculations are $f(\mathbf{a}_1)=3$, the minimum and $f(\mathbf{a}_1)=27$ the maximum value.

11. Find the shortest distance from the origin to $x^2+3xy+y^2=4$.

12. Find the shortest distance from $(0,0,1)^T$ to $y^2+x^2+4xy=4$ in the x-y plane.

Solution This problem is in $\mathbb{R}^3$ even though $y^2+x^2+4xy=4$ appears to be in $\mathbb{R}^2$. The general point of $\mathbb{R}^3$ on $y^2+x^2+4xy=4$ is $(x,y,0)^T$. The (square of the) distance of this point from $(0,0,1)^T$ is x^2+y^2+1. So we need minimise x^2+y^2+1 subject to $y^2+x^2+4xy=4$. Lagrange multipliers give

$$2x=\lambda(2x+4y) \quad \text{and} \quad 2y=\lambda(2y+4x).$$

Rearrange as $(1-\lambda)x=2\lambda y$ and $(1-\lambda)y=2\lambda x$. Then

$$(1-\lambda)^2x=(1-\lambda)2\lambda y=4\lambda^2x.$$

So **either** $x=0$ **or** $(1-\lambda)^2=4\lambda^2$.

- If $x=0$ in $y^2+x^2+4xy=4$ then $y=\pm2$. So we get two critical points

$$(0,2,0)^T \quad \text{and} \quad (0,-2,0)^T.$$

- If $(1-\lambda)^2=4\lambda^2$ then **either** $1-\lambda=2\lambda$ **or** $1-\lambda=-2\lambda$.

  * If $1-\lambda=2\lambda$, i.e. $\lambda=1/3$, then $(1-\lambda)x=2\lambda y$ implies $x=y$. In $y^2+x^2+4xy=4$ this leads to $6x^2=4$, so $x=\pm\sqrt{2/3}$. This gives two more critical points

$$\left(\sqrt{2/3},\,\sqrt{2/3},\,0\right)^T \quad \text{and} \quad \left(-\sqrt{2/3},\,-\sqrt{2/3},\,0\right)^T.$$

  * If $1-\lambda=-2\lambda$, i.e. $\lambda=-1$, then $(1-\lambda)x=2\lambda y$ implies $x=-y$. In $y^2+x^2+4xy=4$ this leads to $-2x^2=4$ which has no real roots and we get no more critical points.

The last two critical points give the minimal distance, 4/3.

13. A cylindrical can (with top and bottom) has volume V. Subject to this constraint, what dimensions give it the least surface area?

Idea of solution If the cylinder of height h and radius r the area is $2\pi rh + 2\pi r^2$ and volume $\pi r^2 h$. So the essence of the question is to minimise $rh + r^2$ subject to $\pi r^2 h = V$.

Solution Define $g(h, r) = \pi r^2 h - V$ and $f(r, h) = rh + r^2$. The set of $(h, r)^T : g(h, r) = 0$ is closed but not bounded. But f is continuous and bounded below by 0. Thus it will have a minimum value.

The Jacobian matrix is $Jg(h, r) = (\pi r^2, 2\pi rh)$. This is only **not** of full rank if $r = 0$ but this does not satisfy $g(h, r) = 0$ for any h. Hence the Jacobian matrix is of full-rank and we can apply Lagrange's method to find $\lambda : \nabla f(h, r) = \lambda \nabla g(h, r)$. That is

$$r = \lambda \pi r^2 \quad \text{and} \quad h + 2r = \lambda 2r\pi h,$$

along with $g(h, r) = 0$.

From $r = \lambda \pi r^2$ we have **either** $r = 0$, but we saw above that this was impossible, **or** $1 = \lambda \pi r$. In the second equation this gives $h + 2r = 2h$, i.e. $h = 2r$. (The height of the cylinder equals the diameter of the base.) In $g(h, r) = 0$ this gives $2\pi r^3 = V$. Then $r = (V/2\pi)^{1/3}$, $h = 2\,(V/2\pi)^{1/3}$ and the surface area is

$$3\,(2\pi)^{1/3}\,V^{2/3}.$$

14. Find the nearest point on the ellipse $x^2 + 2y^2 = 1$ to the line $x + y = 4$.

Idea of solution If $(x, y)^T$ is a point on the ellipse and $(u, v)^T$ a point on the line then $(x - u)^2 + (y - v)^2$ is the square of the distance between the two points. So need to minimise $(x - u)^2 + (y - v)^2$ subject to $x^2 + 2y^2 = 1$ and $u + v = 4$.

15. How close does the intersection of the planes $v + w + x + y + z = 1$ and $v - w + 2x - y + z = -1$ in $\mathbb{R}^5$ come to the origin?

Idea of solution To minimise $v^2 + w^2 + x^2 + y^2 + z^2$ (the square of the distance of $(v, w, x, y, z)^T$ from the origin) subject to $v + w + x + y + z = 1$ and $v - w + 2x - y + z = -1$. The answer is $\sqrt{612}/36$.

Solution Let

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} v + w + x + y + z - 1 \\ v - w + 2x - y + z + 1 \end{pmatrix}.$$

for $\mathbf{x} = (x, y, z, v, w)^T$. Then

$$J\mathbf{g}(\mathbf{x}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 1 & -1 \end{pmatrix}$$

which is of full-rank. So we can apply Lagrange's method, solving $\nabla f(\mathbf{x}) = \lambda \nabla g^1(\mathbf{x}) + \mu \nabla g^2(\mathbf{x})$ for some $\lambda, \mu \in \mathbb{R}$ along with $g^1(\mathbf{x}) = 0$ and $g^2(\mathbf{x}) = 0$. The first condition leads to

$$\begin{aligned} 2v &= \lambda + \mu, \\ 2w &= \lambda - \mu, \\ 2x &= \lambda + 2\mu, \\ 2y &= \lambda - \mu \\ 2z &= \lambda + \mu. \end{aligned}$$

From these we see that $z = v$ and $y = w$. Substituted into $g^1(\mathbf{x}) = 0$ and $g^2(\mathbf{x}) = 0$ we have 5 equations in 5 unknowns:

$$\begin{aligned} 2v &= \lambda + \mu, \\ 2w &= \lambda - \mu, \\ 2x &= \lambda + 2\mu, \\ 2v + 2w + x &= 1, \\ 2v - 2w + 2x &= -1 \end{aligned}$$

The first two give $2v + 2w = 2\lambda$. The second and third $4w + 2x = 3\lambda$. Thus we have three equations in three unknowns:

$$\begin{aligned} 3v - w - 2x &= 0, \\ 2v + 2w + x &= 1, \\ 2v - 2w + 2x &= -1. \end{aligned}$$

In matrix form

$$\begin{pmatrix} 3 & -1 & -2 \\ 2 & 2 & 1 \\ 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} v \\ w \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Thus

$$\begin{pmatrix} v \\ w \\ x \end{pmatrix} = \frac{1}{36}\begin{pmatrix} 6 & 6 & 3 \\ -2 & 10 & -7 \\ -8 & 4 & 8 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{36}\begin{pmatrix} 3 \\ 17 \\ -4 \end{pmatrix}.$$

Then

$$\begin{aligned} f(\mathbf{x}) &= v^2+w^2+x^2+y^2+z^2 = 2v^2+2w^2+x^2 \\ &= \frac{1}{36^2}\left(2\times 3^2+2\times 17^2+(-4)^2\right) \\ &= \frac{612}{36^2}. \end{aligned}$$

We have minimised the square of the distance, so the minimum distance is $\sqrt{612}/36$.

16. Let $x_1, ..., x_5$ be 5 positive numbers. Maximise their product subject to the constraint that $x_1+2x_2+3x_3+4x_4+5x_5 = 300$.

Solution Let $f(\mathbf{x}) = x_1x_2x_3x_4x_5$ and $g(\mathbf{x}) = x_1+2x_2+3x_3+4x_4+5x_5-300$ for $\mathbf{x} = (x_1,\, x_2,\, x_3,\, x_4,\, x_5)^T \in \mathbb{R}^5$. First, $Jg(\mathbf{x}) = (1,2,3,4,5) \neq \mathbf{0}$ and so we can apply Lagrange's method. This means solving $\nabla f(\mathbf{x}) = \lambda\nabla g(\mathbf{x})$ for some $\lambda \in \mathbb{R}$ along with $g(\mathbf{x}) = 0$ and $x_i > 0$ for $1 \le i \le n$. That is,

$$\begin{aligned} x_2x_3x_4x_5 &= \lambda \\ x_1x_3x_4x_5 &= 2\lambda \\ x_1x_2x_4x_5 &= 3\lambda \\ x_1x_2x_3x_5 &= 4\lambda \\ x_1x_2x_3x_4 &= 5\lambda, \end{aligned} \tag{4}$$

with $g(\mathbf{x}) = 0$ and $x_i > 0$ for $1 \le i \le n$. From (4) we see that

$$\lambda x_1 = 2\lambda x_2 = 3\lambda x_3 = 4\lambda x_4 = 5\lambda x_5. \tag{5}$$

If $\lambda = 0$ then, from (4), at least one $x_i = 0$ when $f(\mathbf{x}) = 0$. Presumably we can find larger values for $f(\mathbf{x})$ so assume $\lambda \neq 0$. Then from (5),

$$x_1 = 2x_2 = 3x_3 = 4x_4 = 5x_5.$$

For this $\mathbf{x}$ we have

$$\begin{aligned} g(\mathbf{x}) &= x_1+2x_2+3x_3+4x_4+5x_5-300 \\ &= 5x_5+5x_5+5x_5+5x_5+5x_5-300 \\ &= 25x_5-300. \end{aligned}$$

The requirement $g(\mathbf{x}) = 0$ gives $x_5 = 12$. Thus

$$x_1 = 60,\ x_2 = 30,\ x_3 = 20 \text{ and } x_4 = 15.$$

At this point $\mathbf{x} = (60,\ 30,\ 20,\ 15,\ 12)^T$ we find that $f(\mathbf{x}) = 6480000$.

17. Find the distance from the point $(10, 1, -6)$ to the intersection of the planes $x + y + 2z = 5$ and $2x - 3y + z = 12$.

Solution To minimise $(x-10)^2 + (y-1)^2 + (z+6)^2$ subject to

$$x + y + 2z = 5 \text{ and } 2x - 3y + z = 12. \tag{6}$$

The Jacobian matrix of this level set,

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & -3 & 1 \end{pmatrix},$$

is of full-rank and so we can apply the method of Lagrange multipliers. This means solving

$$\begin{pmatrix} 2(x-10) \\ 2(y-1) \\ 2(z+6) \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix},$$

with $\lambda, \mu \in \mathbb{R}$ along with (6).

$$\begin{aligned} 2(x-10) &= \lambda + 2\mu && \text{(a)}, \\ 2(y-1) &= \lambda - 3\mu && \text{(b)}, \\ 2(z+6) &= 2\lambda + \mu && \text{(c)}. \end{aligned}$$

Then $3\,\text{(a)} + 2\,\text{(b)}$ and $\text{(b)} + 3\,\text{(c)}$ give

$$\begin{aligned} 6(x-10) + 4(y-1) &= 5\lambda, \\ 6(z+6) + 2(y-1) &= 7\lambda. \end{aligned}$$

Remove λ and rearrange to $7x + 3y - 5z = 103$. We thus have

$$\begin{aligned} x + y + 2z &= 5, \\ 2x - 3y + z &= 12, \\ 7x + 3y - 5z &= 103. \end{aligned}$$

Solve. One way is to write it as

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & -3 & 1 \\ 7 & 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \\ 103 \end{pmatrix}.$$

The inverse of the matrix is

$$\frac{1}{83} \begin{pmatrix} 12 & 11 & 7 \\ 17 & -19 & 3 \\ 27 & 4 & -5 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{83} \begin{pmatrix} 12 & 11 & 7 \\ 17 & -19 & 3 \\ 27 & 4 & -5 \end{pmatrix} \begin{pmatrix} 5 \\ 12 \\ 103 \end{pmatrix} = \begin{pmatrix} 11 \\ 2 \\ -4 \end{pmatrix}.$$

Therefore, the nearest point on line is $(11, 2, -4)^T$ and the.distance is $\sqrt{6}$.

18. If a and b are positive numbers find the maximum and minimum values of $(xv - yu)^2$ subject to the constraints $x^2 + y^2 = a^2$ and $u^2 + v^2 = b^2$.

Geometrically Consider two concentric circles with centre the origin, of radius a and b. Let $\mathbf{x} = (x, y)^T$ be a point on the circle of radius a and $\mathbf{u} = (u, v)$ a point on the circle of radius b. Look upon $\mathbf{x}$ and $\mathbf{u}$ as vectors based at the origin. Then $|xv - yu| = |\mathbf{x} \wedge \mathbf{u}|$, which represents the area between the vectors $\mathbf{x}$ and $\mathbf{u}$. It is the case that this is minimised when $\mathbf{x}$ and $\mathbf{u}$ lie in the same direction, for the area will be zero. It doesn't seem unreasonable that the maximum is when $\mathbf{x}$ and $\mathbf{u}$ are orthogonal in which case $|\mathbf{x} \wedge \mathbf{u}| = |\mathbf{x}|\,|\mathbf{u}| = ab$. To prove this we might note that whatever $\mathbf{x}$ and $\mathbf{u}$ are, we can rotate the situation so that $\mathbf{u}$ lies along the x-axis, i.e. $\mathbf{u} = (b, 0)$. Then the problem reduces to one of finding the extrema of y^2b^2 subject to $x^2 + y^2 = a^2$.

19. Find the dimensions of the box parallel to the axes of maximum volume given that the surface area is $10m^2$.

Idea of solution If x, y and z are the lengths of the sides of the box then the volume is xyz and the surface area $2\,(xy + yz + xz)$. So maximise xyz subject to $xy + yz + xz = 5$.

Solution Let $f(\mathbf{x}) = xyz$ and $S(\mathbf{x}) = xy + yz + xz - 5$ for $\mathbf{x} \in \mathbb{R}^3$. Physical constraints imply that $x > 0$, $y > 0$ and $z > 0$ Our problem is to determine the maximum of $f(\mathbf{x})$ subject to $S(\mathbf{x}) = 0$ and $x > 0$, $y > 0$ and $z > 0$. The gradient vectors are

$$\nabla f(\mathbf{x}) = (yz, xz, xy)^T \text{ and } \nabla S(\mathbf{x}) = (y+z, x+z, x+y)^T.$$

Note first that, $\nabla S(\mathbf{x}) = \mathbf{0}$, if, and only if, $\mathbf{x} = \mathbf{0}$, which does not satisfy $S(\mathbf{x}) = 0$. So we can apply the method of Lagrange multiplies, which requires solving $\nabla f(\mathbf{x}) = \lambda \nabla S(\mathbf{x})$ for some $\lambda \in \mathbb{R}$ along with $S(\mathbf{x}) = 0$. That is

$$\begin{aligned} yz &= \lambda (y+z), \\ xz &= \lambda (x+z), \\ xy &= \lambda (x+y). \end{aligned}$$

If $\lambda = 0$ then $yz = xz = xy = 0$. Adding together we see that $S(\mathbf{x}) = -5 \neq 0$. So we have $\lambda \neq 0$.

Multiply by the appropriate factor to get

$$\begin{aligned} xyz &= \lambda (xy+xz), \\ xyz &= \lambda (xy+yz), \\ xyz &= \lambda (xz+yz). \end{aligned}$$

Since $\lambda \neq 0$ we can divide by λ and deduce that $xy + xz = xy + yz = xz + yz$, i.e. $yz = xz = xy$. Since no term is 0 we find that $x = y = z$. In $S(\mathbf{x}) = 5$ this leads to $3x^2 = 5$, i.e. $x = (5/3)^{1/2}$. Then the maximal volume is $(5/3)^{3/2}$.